PROPAGATION OF WAVE FILM TYPE OSCILLATIONS OF QUANTIZED THICKNESS

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The present paper concerns formal solutions of the wave equation and of elasticity theory equations which differ substantially from zero only in the neighborhood of some surface. Physically, such solutions correspond to a wave field whose intensity is markedly different from zero only near this surface. The reflection and refraction of such waves can be considered.

The frequency ω is assumed to be high. All of the arguments to follow are based on the parabolic equation method [1].

1. The scalar case. Derivation of the parabolic equation. Let a wave process be described by a wave equation with the variable velocity b,

$$\frac{1}{b^2(M)}U_{tt} - \Delta U = 0, \quad M = M(x, y, z) \qquad \left(\Delta = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \quad (1.1)$$

Further, in the neighborhood of some smooth surface S let the function U be of the form

$$U \sim \exp\{-i\omega[t - \tau(M)]\} V(M, \omega), \quad \omega \to \infty \qquad (1.2)$$

Here V is a function which differs markedly from zero only near S and varies slowly as compared with the factor $\exp\{-i\omega[t - \tau(M)]\}$, and $\tau(M)$ is some function of the point $M(x, y, z)$.

Examining the case where S coincides with the plane $z \equiv z_0$ and where

$$b(M) = b(z), \quad b'(z_0) = 0, \quad b''(z_0) > 0, \quad \tau(M) = \frac{x}{b(z_0)}, \quad V = V(z, \omega)$$

we readily see that solutions concentrated in the neighborhood of $z = z_0$ do, in fact exist; the domain where V is markedly different from zero is of the form

$$z - z_0 = O\left(\frac{1}{\sqrt{\omega}}\right), \quad \frac{\partial^m}{\partial z^m}V = O(\omega^{m/2}), \quad \text{if} \quad V = O(1)$$

By analogy with this special case, we can assume here that the domain where $V \sim O(1)$ is of the form

$$|\nu| = O\left(\frac{1}{\sqrt{\omega}}\right)$$

Here ν is the distance along the normal from S taken together with its sign. Further, in the "boundary layer" $|\nu| = O(\omega^{-1/2})$ we set

$$\frac{\partial^{k+r+m}}{\partial s_1{}^k \partial s_2{}^k \partial \nu^m} V = O(\omega^{1/2 m})$$

Here $\partial/\partial s_1$, $\partial/\partial s_2$ represents differentiations tangentially to S. Let us impose on S a coordinate grid consisting of the curves $\tau(M) = \text{const}$, $M \in S$ and the curves orthogonal to them. We shall assume that the resulting coordinate grid α, τ is regular. In the neighborhood of S we introduce curvilinear coordinate in accordance with Formula

$$\mathbf{X} = \mathbf{X}(\alpha, \tau) + \nu \mathbf{n}(\alpha, \tau), \ \mathbf{X} = \mathbf{X}(x, y, z) \tag{1.3}$$

Here $\mathbf{X} = \mathbf{X}(\alpha, \tau)$ is the parametric representation of the surface S, $\mathbf{n} = \mathbf{n}(\alpha, \tau)$ is the unit normal to S at the point α, τ; ν is the distance of the point $(x, y, z) = \mathbf{X}$ from S. We shall assume that ν is of a different sign to each side of S.

We shall make use of the following expression for the Laplacian; if in the curvilinear coordinates q^1, q^2, q^3, an element ds is given by

$$ds = \sqrt{G_{ij}\, dq^i\, dq^j} \tag{1.4}$$

then

$$\Delta = \frac{1}{\sqrt{G}} \frac{\partial}{\partial q^i} \left(G^{ij} \sqrt{G} \frac{\partial (\cdots)}{\partial q^s} \right), \qquad G = \det \| G_{ij} \| \tag{1.5}$$

The matrix $\|G^{ij}\|$ is inverse to the matrix $\|G_{ij}\|$.

In the case of the coordinates α, τ, ν the matrix

$$\|G_{ij}\| = \left\| \begin{matrix} G_{\alpha\alpha} & G_{\alpha\tau} & 0 \\ G_{\tau\alpha} & G_{\tau\tau} & 0 \\ 0 & 0 & 0 \end{matrix} \right\|$$

$$\begin{gathered} G_{\alpha\alpha} = |\mathbf{X}_\alpha|^2 + 2(\mathbf{X}_\alpha, \mathbf{n}_\alpha)\nu + \nu^2 \mathbf{n}_\alpha{}^2, \quad G_{\tau\tau} = |\mathbf{X}_\tau|^2 + 2(\mathbf{X}_\tau, \mathbf{n}_\tau)\nu + \mathbf{n}_\tau{}^2 \nu^2 \\ G_{\alpha\tau} = G_{\tau\alpha} = (\mathbf{X}_\alpha, \mathbf{X}_\tau) + [(\mathbf{X}_\alpha, \mathbf{n}_\tau) + (\mathbf{X}_\tau, \mathbf{n}_\alpha)]\nu + (\mathbf{n}_\alpha, \mathbf{n}_\tau)\nu^2 \\ G_{\alpha\nu} = G_{\tau\nu} = G_{\nu\alpha} = G_{\nu\tau} = 0, \qquad G_{\nu\nu} = 1 \end{gathered} \tag{1.6}$$

Substituting Expression (1.2) into Equation (1.1) we obtain

$$\Delta (V e^{i\omega\tau}) + \frac{\omega^2}{b^2(M)} V e^{i\omega\tau} = 0 \tag{1.7}$$

Let us make use of Formulas (1.5) and (1.6), assuming that

$$\nu = O\left(\frac{1}{\sqrt{\omega}}\right), \qquad \frac{\partial^{k+l+m}}{\partial \tau^k \partial \alpha^l \partial \nu^m} V = O(\omega^{m/2})$$

In the left-hand side of Formula (1.7) the principal terms are those of orders ω^2, $\omega^{3/2}$, ω. Equating terms with ω^2 and $\omega^{3/2}$, to zero, we obtain, respectively

$$\frac{1}{b^2}\bigg|_{\nu=0} - G^{\tau\tau}\big|_{\nu=0} = 0, \qquad -2\frac{(\mathbf{X}_\tau, \mathbf{n}_\tau)}{|\mathbf{X}_\tau|^2} - \frac{\partial}{\partial \nu}\left(\frac{1}{b^2}\right)\bigg|_{\nu=0} = 0$$

These equations mean that on the surface S the lines along which the parameter τ varies (i.e. the lines $\alpha = \text{const}$) are rays, i.e. extrema of the Fermat integral $\int b^{-1} ds$. Thus, the surface S is a "weave" of rays— something which might have been expected.

Finally, we equate to zero the terms of order ω to obtain

$$\frac{\partial^2 V}{\partial \nu^2} + 2i\omega G^{\tau\tau} V_\tau + \frac{i\omega}{\sqrt{G}} \frac{\partial}{\partial \tau} (\sqrt{G}\, G^{\tau\tau}) V + \Phi(\alpha, \tau)\, \nu^2 \omega^2 V = 0 \qquad (1.8)$$

where

$$\Phi(\alpha, \tau) = -\frac{b_{\nu\nu}}{b^3}\bigg|_{\nu=0} + \frac{4(\mathbf{X}_\alpha, \mathbf{n}_\alpha)(\mathbf{X}_\tau, \mathbf{n}_\tau) + 3|\mathbf{X}_\alpha|^2 |\mathbf{n}_\tau|^2}{|\mathbf{X}_\alpha|^2 |\mathbf{X}_\tau|^2}$$

2. The scalar case. Solution of the parabolic equation. As is common in problems of this type, we substitute variables:

$$\nu = \sqrt{\omega}\, \psi(\alpha, \tau)\, \zeta, \qquad V = \sqrt{\sqrt{G}\, G^{\tau\tau}} \exp(-i\psi_\tau / \psi^3 G^{\tau\tau} \zeta^2)\, W$$

where ψ is a function to be determined.

For the new required function we obtain (2.1)

$$W_{\zeta\zeta} + \zeta^2 \left[\left(\frac{\psi_\tau}{\psi^3} G^{\tau\tau} \right)^2 + \frac{G^{\tau\tau}}{\psi^2} \left(\frac{\psi_\tau}{\psi^3} G^{\tau\tau} \right)_\tau - \frac{\Phi(\alpha, \tau)}{\psi^4} \right] W = -\frac{2iG^{\tau\tau}}{\psi^2} W_\tau + \frac{i\psi_\tau}{\psi^3} G^{\tau\tau} W$$

In order to guarantee separability of the variables it is sufficient for us to equate the expression in square brackets to a constant. This yields a linear second-order equation for determining $\psi(\alpha, \tau)$.

Let this constant (*) be -1:

$$\left(\frac{\psi_{\tau\tau}}{\psi^3} G^{\tau\tau} \right)^2 + \frac{G^{\tau\tau}}{\psi^2} \left(\frac{\psi_\tau}{\psi^2} G^{\tau\tau} \right)_\tau - \frac{\Phi}{\psi^4} = -1 \qquad (2.2)$$

The substitution

$$\psi = \frac{1}{G^{\tau\tau} \beta(\alpha, \tau)}$$

allows this equation to be written as

$$\frac{\beta''}{\beta} + F(\alpha, \tau) = \frac{1}{\beta^4}, \qquad F(\alpha, \tau) = \frac{(G^{\tau\tau})_{\tau\tau}}{G^{\tau\tau}} - \frac{\Phi(\alpha, \tau)}{(G^{\tau\tau})^2} \qquad (2.3)$$

The general solution of this Eq. is

$$\beta(\tau) = \left(\sum_{i,k=1}^{2} a_{ik} y_i(\tau) y_k(\tau) \right)^{1/2} \qquad (2.4)$$

Here y_1, y_2 is any pair of linearly independent solutions of Eq.

$$y'' + Fy = 0$$

It can be shown that the determinant of the positively defined matrix $\|a_{ik}\|$ is related to the Wronskian of the solution (y_1, y_2) by Expression

$$\frac{\det \|a_{ik}\|}{(y_1 y_2' - y_1' y_2)^2} = 1$$

When relation (2.2) is fulfilled the variables in Equation (2.1) become separable and we obtain

$$W = Z(\zeta) \sqrt{\omega} \exp\left(-\frac{i}{2} \lambda \int G^{\tau\tau} \psi^2 \, d\tau \right)$$

Here $Z(\zeta)$ is the solution of

$$Z'' + (\lambda - \zeta^2)\, Z = 0$$

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*) The assumption that $C = -1$ does not limit the generality of our discussion, since for $C > 0$ the solution is not concentrated in the neighborhood of $\nu = 0$, while $C < 0$ ($C \neq -1$) drops outof the final formulas.

The equation for Z is Weber's classical equation. It has a solution which tends to zero at infinity if and only if $\lambda = 2n + 1, n = 0, 1, 2, \ldots$ In this case

$$Z = Zn(\zeta) = \exp(-{}^1/_2\zeta^2) H_n(\zeta) \tag{2.5}$$

where $H_n(\zeta)$ is the n-th Hermite polynomial.

Collecting the above formulas, we obtain Expression

$$U = \chi(\alpha) \frac{b}{|X_\alpha|} \sqrt{\psi} \exp\left[-\frac{i(2n+1)}{2}\int b^2\psi^2\, d\tau - \frac{i\psi_\tau}{b^2\psi^2}\zeta^2\right] \times$$
$$\times \exp\frac{-\zeta^2}{2} H_n(\zeta) e^{-i\omega(t-\tau)} \qquad (n = 0, 1, 2, \ldots) \tag{2.6}$$
$$\zeta = \frac{\nu}{\sqrt{\omega}\,\psi(\alpha, \tau)}, \qquad \psi = \frac{b^2}{\beta(\alpha, \tau)}$$

Here $\varkappa(\alpha)$ is an arbitrary function of α, and β is given by Formula (2.4). Solutions (2.6) are precisely the quantized-thickness wave film type solutions we have been seeking.

For $|\zeta| \leqslant \sqrt{2n+1}$ the function (2.5) oscillates; for $|\zeta| > \sqrt{2n+1}$ — it tends monotonously to zero.

Thus, $|\zeta| = \sqrt{2n+1}$ is the arbitrary thickness of the domain or "wave film" where the oscillations occur. Substituting for ζ its expression, we find that the thickness of the wave film is "quantized" and can assume values of the form

$$|\nu| = \sqrt{2n+1}\,\omega^{-1/2} \frac{1}{\psi(\alpha, \tau)} \qquad (n = 0, 1, 2, \ldots) \tag{2.7}$$

3. The energy conservation law. Reflection and refraction of wave film type waves. 1. For waves of this type, the energy propagates along the rays in the first approximation.

Let us elaborate this statement. In the case where the wave process is described by Equation (1.1), the energy density is of the form

$$\frac{1}{2}\frac{1}{b^2(M)} U_t\overline{U}_t + \frac{1}{2}(U_x\overline{U}_x + U_y\overline{U}_y + U_z\overline{U}_z)$$

The bar denotes the complex conjugate.

Let us consider the energy dE confined in the small volume

$$\alpha_0 < \alpha \leqslant \alpha_0 + d\alpha, \quad t \leqslant \tau \leqslant t + d\tau, \quad -\nu_0 \leqslant \nu \leqslant \nu_0,\ \nu_0 = O\ (1) \tag{3.1}$$

Here t is time. In computing dE we make use of Expression (2.6). In the small moving volume (2.8) as $\omega \to \infty$ we consider only the principal energy term, which gives us

$$dE = |\chi(\alpha)|^2 \frac{b}{|X_\alpha|^2} \psi\omega^2 \frac{1}{b^2} \int_{-\nu_0}^{\nu_0} \exp -\frac{\zeta^2}{2} H_n(\zeta)\, d\nu\, |X_\tau|\, d\tau\, |X_\alpha|\, d\alpha$$

Making use of the fact that $\zeta = \sqrt{\omega}\,\psi(\alpha, \tau)\,|\,\nu$, $|X_\tau| = b$ and replacing the integration limits by $\pm\infty$, we arrive at the Formula

$$dE = |\chi(\alpha)|^2 \int_{-\infty}^{+\infty} \exp\frac{-\zeta^2}{2} H_n(\zeta)\, d\zeta\, \omega^{3/2}\, d\tau\, d\alpha = |\chi(\alpha)|^2 \Delta\tau\, \Delta\alpha n!\, \sqrt{2\pi}$$

This expression depends solely on the ray. Thus, the energy in moving volume (2.8) does not change when this volume moves along rays with the wave velocity b.

2. Waves of the wave film type can be reflected and refracted. The reflected and refracted waves are defined unambiguously and are also waves of the wave film type.

Let some surface Σ be struck by a wave of the wave film type. We assume that Σ and the surface S intersect at nonzero angles. If the classical boundary conditions $U|_\Sigma = 0$ for $[\partial u / \partial n]_\Sigma = 0$, are fulfilled on the surface Σ, then one wave film will not satisfy a boundary condition of this type.

The reflected wave will be sought on the basis of the fact that the sum of the incident and reflected waves satisfies the boundary condition, i. e. that

$$U_1 + U_2|_\Sigma = \sum_{i,j=1}^{2} \chi_j(\alpha_j) \sqrt{\frac{b}{|X_{j\alpha}|}} \sqrt{\psi_j} \exp\left[-\frac{i2n+1}{2} \int b^2\psi_j^2 d\tau_1 - \right.$$
$$\left. - \frac{i\psi_{j\tau}}{b^2\psi_j^3} \zeta_j^2\right] \exp \frac{-\zeta_j^2}{2} H_{n_j} \exp\left[-i\omega(t - \tau j)\right]\Big|_\Sigma = 0$$

The quantities referring to the incident and reflected waves are denoted by the subscript 1 and 2 respectively.

Let us limit ourselves to the boundary condition $U|_\Sigma = 0$; the case of the boundary condition $[\partial U / \partial n]_\Sigma = 0$ can be analyzed in similar fashion.

First, in order for the Expression $(U_1 + U_2)|_\Sigma$ to be capable of going to zero, it is necessary that the most rapidly oscillating terms coincide.

Let σ be the intersection of S and Σ and let ℓ be the family of lines on Σ orthogonal to σ. We shall assume that one and only one line ℓ passes through each point on Σ near σ.

Coincidence in principal terms of the most rapidly oscillating factors in the expressions for U_1 and U_2 requires that

$$\tau_1|_\sigma = \tau_2|_\sigma, \qquad \frac{\partial \tau_1}{\partial l}\Big|_\sigma = \frac{\partial \tau_2}{\partial l}\Big|_\sigma$$

Here $\partial \tau_j / \partial \ell$ is the derivative of τ_j along the arc of the curve ℓ.

These conditions are equivalent to the fact that the surfaces S_j of the incident and reflected wave films intersect Σ at equal angles, i. e. the angle of incidence of wave films is equal to their angle of reflection.

Thus, the factors $\exp[-{}^1/_2\zeta_j^2]H_{nj}(\zeta_j)$ $(j = 1, 2)$ coincide if $n_1 = n_2$ and $\partial\zeta_1 / \partial l = \partial\zeta_2 / \partial l|_\sigma$ (Our statements are valid to within the principal terms.)
Eq. $\partial\zeta / \partial l = \partial\zeta_2 / \partial l|_\sigma$ is equivalent to

$$\psi_2(\alpha, \tau) = \psi(\alpha, \tau)|_\sigma \tag{3.2}$$

Next, we equate Expressions

$$(-i)\frac{\zeta_2^2}{b^2\psi_2^3}\frac{\partial\psi_2}{\partial\tau} + \frac{i}{2}\omega\frac{\partial^2\tau_2}{\partial l^2}l^2 \approx (-i)\frac{\zeta_1^2}{b^2\psi_1^3}\frac{\partial\psi_1}{\partial\tau} + \frac{i}{2}\omega\frac{\partial^2\tau_1}{\partial l^2}l^2 \tag{3.3}$$

Here ℓ is the arc length along the curve ℓ as measured from σ. Making use of the fact that there exist the finite limits

$$\lim\ (\zeta_j^2 / \omega l^2) \quad l \to 0$$

we can readily find $\partial\psi_2/\partial\tau|_\sigma$ from (3. 3).

Thus, the function ψ_2 and its first derivative are known on σ. We recall that ψ_2 is the solution of an ordinary differential equation of the (2. 2) type along the reflected ray. Specification of the initial conditions for this equation determines its solution, i. e. the function ψ_2, unambiguously.

Now only those factors which are constant in the first approximation throughout the thickness of the film differ in Expression for U_1 and U_2. Equating them with the opposite signs, we determine $X_2(\alpha)$ unambiguously. The refraction of wave films can be considered in analogous fashion.

4. Propagation of elastic oscillations of the wave film type. Without making any fundamental alterations, we can use the same technique to find waves of the "wave film" type for transverse elastic oscillations.

We introduce the same coordinates α, τ, ν as in Secs. 1 and 2. We assume from the start that the wave film surface S is "blanketed" by rays.

Let an element of length in the curvilinear coordinates q^1, q^2, q^3, be given by (1.4). Further, let $\varphi^1, \varphi^2, \varphi^3$ be components of the displacement vector in these coordinates; more precisely, if elastic deformation shifts $(\cdot) M(q^1, q^2, q^3)$ to the position $M(q^{1\prime}, q^{2\prime}, q^{3\prime})$, then $\varphi^j = q^{j'} - q^j$ $(j = 1, 2, 3)$. The system of elasticity theory equations in the coordinates $q^{j\prime}$ is of the form

$$\sigma_{i'j'} G^{ii'} G^{jj'} \sqrt{G}\, \frac{\partial G_{ij}}{\partial q^s} + \frac{\partial}{\partial q^j} (\sigma_{ss'} G^{jj'} \sqrt{G}) + \frac{\partial}{\partial q^i} (\sigma_{i's} G^{ii'} \sqrt{G}) -$$
$$- 2\rho G_{i's} \sqrt{G}\, \frac{\partial^2 \varphi^i}{\partial t^2} = 0 \qquad (s = 1, 2, 3) \tag{4.1}$$

The recurring indices denote summation from 1 to 3.

Here σ_{ij} are the components of the stress tensor related to the displacement vector by the expressions (Hooke's law)

$$\sigma_{i'j'} = \frac{\lambda}{\sqrt{G}} \frac{\partial}{\partial q^s} (\varphi^s \sqrt{G})\, G_{i'j'} + \mu \left[\frac{\partial G_{i'j'}}{\partial q^s} \varphi^s + G_{si} \frac{\partial \varphi^s}{\partial q^{j'}} + G_{j's} \frac{\partial \varphi^s}{\partial q^{i'}} \right] \tag{4.2}$$

We shall assume that the Lamé parameters λ and μ are smooth functions of the coordinates and that the displacement vector is of the form

$$\mathbf{U} = V e^{-i\omega[t - \tau(M)]}\, \mathbf{l}_0, \qquad |\mathbf{l}_0| = 1, \ \mathbf{l}_0 \perp \nabla \tau, \quad \mathbf{l}_0 \| S$$

(transverse waves); hence,

$$\varphi^2 = |\mathbf{X}_\alpha|\, V e^{-i\omega[t - \tau(M)]}, \qquad \varphi^1 = \varphi^3 = 0 \tag{4.3}$$

Let us assume that the derivatives of V are of the order

$$\frac{\partial^{r+k+m}}{\partial \alpha^r\, \partial r^k\, \partial \nu^m} V = O(\omega^{1/2 m})$$

Substituting the expressions for $\varphi^1, \varphi^2, \varphi^3$ from Formulas (4.3) into Formulas (3.1) and (3.2) (the components of the tensors G_{ij}, G^{ij} are of the form (1.6)), we find that terms of orders ω^2, $\omega^{3/2}$, ω drop out of the first and third equations. More precisely, left-hand side of the first Eq. is of the order $O(1)$, while the third is of the order $O(\omega^{-1/2})$.

The principal terms in the second equation are of order ω. Equating them to zero, we arrive at the parabolic equation

$$V_{\nu\nu} + \frac{2i\omega}{\mu} G^{\tau\tau} V_\tau + \frac{i\omega}{\mu \sqrt{G} G_{\alpha\alpha}} \frac{\partial}{\partial \tau} (G^{\tau\tau} \sqrt{G}\, \mu G_{\alpha\alpha}) V + \left(\omega^2 \Big| \frac{1}{b^2} - G^{\tau\tau} \right) V = 0$$

This equation is solved in exactly the same way as Eq. (1.8). The final Formulas are

$$\mathbf{U} = \sqrt{\frac{b}{|\mathbf{X}_\alpha|}} \sqrt{\frac{\psi}{\mu}} \exp \left[-\frac{i(2n+1)}{2} \int b^2 \psi^2\, d\tau - \frac{i\psi'}{b^2 \psi^3} \right] \times$$
$$\times \exp \frac{-\zeta^2}{2} H_n(\zeta)\, e^{-i\omega(t-\tau)}\, \mathbf{l}_0 \left(\zeta = \frac{\sqrt{\omega}\,\psi(\alpha, \tau)}{\nu}, \ b = \sqrt{\frac{\mu}{\rho}} \right) \tag{4.4}$$

Here H_n is a Hermite polynomial, b is the velocity of the transverse waves, and

$\psi(\alpha, \tau)$ can be found from Eq.

$$\frac{\beta''}{\beta} + F(\alpha, \tau) = \frac{1}{\beta^4}, \qquad \beta = \frac{b^2}{\psi}$$

$$F = \frac{b_{\tau\tau}}{b} - \frac{b_{\nu\nu}}{b^3\mu}\bigg|_{\nu=0} + \frac{4(\mathbf{X}_\alpha, \mathbf{n}_\alpha)(\mathbf{X}_\tau, \mathbf{n}_\tau) + 3|\mathbf{X}_\alpha|^2|\mathbf{n}_\tau|^2}{|\mathbf{X}_\alpha|^2|\mathbf{X}_\tau|^4} \tag{4.5}$$

As in the scalar case, we can consider reflection and refraction of wave films of the type just investigated. For waves of the (3.4) type the energy propagates along "rays" in the same sense as in the scalar case.

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